## DD-2811

## M. A./M. Sc. (Final) EXAMINATION, 2020

MATHEMATICS
(Optional)
Paper Fourth (ii)
(Wavelets)
Time : Three Hours
Maximum Marks : 100
Note : Attempt any two parts from each question. All questions carry equal marks.

Unit-I

1. (a) If the operator $P=P_{0, \in}$ defined by :

$$
(\mathrm{P} f)(x) \equiv \overline{\mathrm{S}(x)}[\mathrm{S}(x) f(x) \pm \mathrm{S}(-x) f(-x)]
$$

then show that P is idempotent, self-adjoint and an orthogonal projection.
(b) Define Multiresolution Analysis. Show that if $g \in \mathrm{~L}^{2}(\mathbf{R})$, then $\{g(.-k): k \in \mathrm{Z}\} \quad$ is $\quad$ an orthonormal system if and only if :

$$
\sum_{k \in \mathrm{Z}}|\hat{g}(\xi+2 k \pi)|^{2}=1 \text { for a.e. } \xi \in \mathbf{R}
$$

(c) Let $r$ be a non-negative integer. Let $\psi$ be a function in $\mathrm{C}^{r}(\mathbf{R})$ such that :

$$
|\psi(x)| \leq \frac{c}{(1+|x|)^{r+1+\epsilon}}
$$

for some $\in>0$, and let $\psi^{(m)} \in \mathrm{L}^{\infty}(\mathbf{R})$ for $m=1,2$, $\ldots . ., r$. If $\left\{\psi_{j, k}: j, k \in \mathbf{Z}\right\}$ is an orthonormal system in $L^{2}(\mathbf{R})$, then show that all moments of $\psi$ upto order $r$ zero; that is :

$$
\int_{\mathrm{R}} x^{m} \psi(x) d x=0
$$

for all $m=0,1,2, \ldots ., r$.

## Unit-II

2. (a) Suppose that $f \in \mathrm{~L}^{2}(\mathbf{R})$ and $\hat{f}$ has a support contained in $\mathrm{I}=(a, b)$, where $b-a \leq 2^{-\mathrm{J}} \pi$ and $\mathrm{I} \cap[-\pi, \pi]=\phi$; then show that for all $j \in \mathrm{Z}:$

$$
\left(\mathrm{Q}_{j} f\right)^{\wedge}(\xi)=\hat{f}(\xi)\left|\hat{\psi}\left(2^{-j} \xi\right)\right|^{2}
$$

a.e. on I.
(b) If $\psi$ is band-limited orthonormal wavelet such that $|\hat{\psi}|$ is continuous at 0 , then show that $\hat{\psi}=0$ a.e. in an open neighbourhood of the origin.
(c) If $f \in \mathrm{~L}^{2}(\mathrm{~T})$, then show that:

$$
\left\{f, \mathrm{U} f, \ldots, \mathrm{U}^{\mathrm{N}} f\right\}
$$

where $\mathrm{U} \equiv \mathrm{U}_{j}$ is an orthonormal system if and only if :

$$
\sum_{l \in Z}\left|\mathrm{~F}[f]\left(n+2^{j} l\right)\right|^{2}=2^{-j}
$$

for $n=0,1,2, \ldots \ldots ., \mathrm{N}=2^{j}-1$.

## Unit-III

3. (a) If $\psi$ is an orthonormal wavelet, then show that :
$\psi\left(2^{n} \xi\right)=\sum_{j=1}^{\infty} \sum_{k \in Z} \hat{\psi}\left(2^{n}(\xi+2 k \pi) \hat{\psi} \overline{\left(2^{j}(\xi+2 k \pi)\right)}\right.$
a.e. for all $n \geq 1$.
(b) Let $\left\{v_{j}: j \geq 1\right\}$ be a family of vectors in a Hilbert space $H$ such that :
(i) $\sum_{n=1}^{\infty}\left\|v_{n}\right\|^{2}=c<\infty$
(ii) $\quad v_{n}=\sum_{m=1}^{\infty}<v_{n} \cdot v_{m}>v_{m}$ for all $n \geq 1$

Let $\mathrm{F}=\overline{\operatorname{span}\left\{v_{j}: j \geq 1\right\}}$, then show that:

$$
\operatorname{dim} \mathrm{F}=\sum_{j=1}^{\infty}\left\|v_{j}\right\|^{2}=c
$$

(c) Define low-pass filter. Let $\mu_{1}, \mu_{2}, \ldots ., \mu_{n}$ be $2 \pi$ periodic functions and set:

$$
\mathrm{M}_{j}=\sup _{\xi \in \mathrm{T}}\left(\left|\mu_{j}(\xi)\right|^{2}+\mid \mu_{j}\left(\xi+\left.\pi\right|^{2}\right)\right.
$$

then show that :

$$
\int_{-2^{n} \pi}^{2^{n} \pi} \prod_{j=1}^{n}\left|\mu_{j}\left(2^{-j} \xi\right)\right|^{2} d \xi<2 \pi \mathrm{M}_{1} \ldots . \mathrm{M}_{n}
$$

## Unit-IV

4. (a) Define frame operator. For any $h \in \mathrm{~L}^{2}(\mathbf{R})$, if $\mathrm{Q} h \in \mathrm{~L}^{2}(\mathrm{R})$ and $p h \in \mathrm{~L}^{2}(\mathbf{R})$, then show that :
(i) $\quad \mathrm{R}(\mathrm{Q} h)(\mathrm{S}, t)=\mathrm{S}(\mathrm{R} h)(\mathrm{S}, t)+\frac{1}{2 \pi i} \frac{\partial}{\partial t}(\mathrm{R} h)(\mathrm{S}, t)$
(ii) $\quad \mathrm{R}(\mathrm{P} h)(\mathrm{S}, t)=-i \frac{\partial}{\partial \mathrm{~S}}(\mathrm{R} h)(\mathrm{S}, t)$
(b) Suppose that:

$$
g \in \mathrm{~L}^{2}(\mathbf{R})
$$

and $\quad g_{m, n}(x)=e^{2 \pi i m x} g(x-n) \quad m, n \in \mathrm{Z}$
If $\left\{g_{m, n}: m, n \in \mathrm{Z}\right\}$ is a frame for $\mathrm{L}^{2}(\mathbf{R})$, then show that either :

$$
\int_{\mathrm{R}} x^{2}|g(x)|^{2} d x=\infty
$$

or

$$
\int_{\mathrm{R}} \xi^{2}|\hat{g}(\xi)|^{2} d \xi=\infty
$$

(c) Show that when $\left\{\mathrm{Q}_{j}: j \in \mathrm{~J}\right\}$ is a frame, $f$ can be reconstructed from the coefficients $<f, \mathrm{Q}_{j}>$ using the dual frame $\left\{\widetilde{\mathrm{Q}}_{j}: j \in \mathrm{~J}\right\}$ and that $f$ is also superposition of $\mathrm{Q}_{j}^{\prime} \mathrm{S}$ with coefficients $<f, \widetilde{\mathrm{Q}}_{j}>$.

## Unit-V

5. (a) If $\mathrm{N}=2^{q}, \mathrm{C}_{\mathrm{N}}=\mathrm{E}_{1} \mathrm{E}_{2} \ldots . . \mathrm{E}_{q}$, where each $\mathrm{E}_{j}$ is an $\mathrm{N} \times \mathrm{N}$ matrix such that each row has precisely two non-zero entries.
(b) Show that $\tilde{y}_{k}$ equals $\frac{1}{2} e^{\frac{\pi i k}{2 N}}$ times the DCT coefficients $\alpha_{k}^{(\mathrm{N})}$ for the function $f$.
(c) Show that the sequence:

$$
\left\{u_{j, k}: 1 \leq k \leq l_{j}-1\right\}
$$

is an orthonormal basis for $\mathrm{E}_{j}$.

