## DD-2804

## M. A./ M. Sc. (Previous) <br> EXAMINATION, 2020

MATHEMATICS
Paper Fourth
(Complex Analysis)
Time : Three Hours
Maximum Marks : 100
Note : Attempt any two parts of each question. All questions carry equal marks.

## Unit-I

1. (a) State and prove Cauchy's Integral formula.
(b) Let $f(z)$ be analytic in the region $|z|<\rho$ and let $z=r e^{i \theta}$ be any point of this region. Then :

$$
f\left(r e^{i \theta}\right)=\frac{1}{2 \pi} \int_{0}^{2 \pi} \frac{\left(\mathrm{R}^{2}-r^{2}\right) f\left(\mathrm{Re}^{i \phi}\right)}{\mathrm{R}^{2}-2 \mathrm{R} r \cos (\theta-\phi)+r^{2}} d \phi
$$

where $R$ is any number such that $0<R<\rho$.
(c) Show that:

$$
e^{\frac{1}{2} c\left(z-\frac{1}{2}\right)}=\sum_{n=-\infty}^{\infty} a_{n} z^{n},
$$

where $a_{n}=\frac{1}{2 \pi} \int_{0}^{2 \pi} \cos (n \theta-c \sin \theta) d \theta, c>0$.

## Unit-II

2. (a) Show that:

$$
\int_{0}^{2 \pi} \frac{d \theta}{a+b \cos \theta}=\frac{2 \pi}{\sqrt{a^{2}-b^{2}}}, a>b>0
$$

(b) Cross-ratios are invariant under a bilinear transformation. Prove.
(c) Show that the transformation $w=\tan ^{2}\left(\frac{\pi}{4} \sqrt{z}\right)$ transforms the interior of the unit circle $|w|=1$ into the interior of a parabola.

## Unit-III

3. (a) If $|z| \leq 1$ and $p \geq 0$, then:

$$
\left|1-\mathrm{E}_{p}(z)\right| \leq|z|^{p+1},
$$

where $\mathrm{E}_{p}(z)$ is elementary factor.
(b) Show that the function:

$$
f_{1}(z)=1+z+z^{2}+z^{3}+\ldots . .+z^{n}+\ldots \ldots
$$

can be obtained outside the circle of convergence of the power series.
(c) State and prove Harnack's inequality.

## Unit-IV

4. (a) If $f(z)$ is analytic within and on the circle $\gamma$ such that $|z|=\mathrm{R}$ and if $f(z)$ has zeros at the points $a_{i} \neq 0, \quad(i=1,2,3, \ldots, m) \quad$ and poles at $\quad b_{j} \neq 0$,
( $j=1,2,3, \ldots, n$ ) inside $\gamma$, multiple zeros and poles being repeated, then :

$$
\left.\begin{array}{rl}
\frac{1}{2 \pi} \int_{0}^{2 \pi} \log \left|f\left(\mathrm{Re}^{i \theta}\right)\right| & d \theta
\end{array}\right)=\log |f(0)|, \quad \begin{aligned}
m & \log \frac{\mathrm{R}}{\left|a_{i}\right|}-\sum_{j=1}^{n} \log \frac{\mathrm{R}}{\left|b_{j}\right|}
\end{aligned}
$$

(b) State and prove Poisson-Jensen formula.
(c) State and prove Hadamard's three circle theorem.

## Unit-V

5. (a) Let $g$ be analytic in $\mathrm{B}(0 ; \mathrm{R}), g(0)=0$, $\left|g^{\prime}(0)\right|=\mu>0$ and $|g(z)| \leq M$ for all $z$, then $g(\mathrm{~B}(0 ; \mathrm{R})) \supset \mathrm{B}\left(0 ; \frac{\mathrm{R}^{2} \mu^{2}}{6 \mathrm{M}}\right)$.
(b) State and prove Schottky's theorem.
(c) Let $\mathrm{F} \in \mathrm{H}(\mathrm{D}-\{0\})$, F be one-to-one in D and

$$
\mathrm{F}(z)=\frac{1}{z}+\sum_{n=0}^{\infty} \alpha_{n} z^{n}(z \in \mathrm{D})
$$

then $\sum_{n=1}^{\infty} n\left|\alpha_{n}\right|^{2} \leq 1$.

